Qualitative Analysis of Continuous Dynamic Systems by Intelligent Numeric Experimentation

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Abstract

This paper describes a program called PHASER that partitions the phase space of a system of two ordinary differential equations into regions of trajectories with equivalent asymptotic behaviors. The boundaries between regions are limit cycles, separatrices, and the unstable manifolds of saddles. PHASER detects these boundaries by numeric simulation, guided by knowledge about dynamics. It determines the behavior near fixed points from the Jacobian of the system, scans outward to find separatrices and limit cycles, and stops upon reaching the boundary of a user-specified region of interest. It analyzes nonhyperbolic fixed points with a Liapunov function or by examining nearby trajectories. It describes the asymptotic behavior of the system in terms of the qualitative properties of the partition, abstracting away the numeric details of the boundary trajectories. The Poincaré-Bendixson theorem, which provides a complete characterization of asymptotic behavior in the plane, enables PHASER to classify every trajectory that it generates. The next challenge is to handle larger systems where the theorem fails.

1 Introduction

Dynamic systems are a key modeling tool in virtually every field of science and engineering. They provide a rich language for describing and analyzing evolving phenomena and artifacts, such as circuits, mechanical devices, chemical reactions, weather patterns, and human physiology. This paper describes a program called PHASER that derives the qualitative behavior of dynamic systems by intelligent numeric simulation. Simulation is necessary because most systems defy complete theoretical analysis. Yet simulation alone is inefficient and unreliable. It requires huge amounts of computation to examine the entire space of input conditions for interesting properties and often derives spurious qualitative behaviors because of round-off errors and other inaccuracies. PHASER ameliorates these problems with theoretical knowledge, both about the specific system under investigation and about general characteristics of systems. It uses this knowledge to guide the simulation and to interpret its output.

The input to PHASER consists of a system of two autonomous continuously differentiable ordinary differential equations and of a bounding box of admissible values for the dependent variables. The joint values of the two variables determine the *state* of the system. Every initial state yields a unique solution to the equations. The output of PHASER is a partition of the states in the bounding box according to the qualitative behaviors of the corresponding solutions. The next two sections explain how PHASER represents states, solutions, and qualitative behaviors. The following sections describe and demonstrate how it derives qualitative behavior by intelligent numeric simulation. The final section contains conclusions and plans for future work.

2 Phase space

PHASER adopts the standard phase space representation of dynamic systems theory. The phase space for a system of two autonomous differential equations

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y)$$

is the Cartesian product of the domains of x and y. Points in phase space represent states of the system. Curves on which the equations are satisfied, called *trajectories,* represent solutions. The topological properties of trajectories characterize the qualitative behavior of solutions. A point trajectory, called a *fixed point,* indicates an equilibrium solution, whereas a closed curve, called a *limit cycle,* indicates a periodic solution. A fixed point or limit cycle is called an *attractor* when all nearby points approach it asymptotically and a *repellor* when all nearby points move away.

The Jacobian matrix

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

determines the behavior of solutions near a fixed point. The fixed point is an attractor when the real parts of both eigenvalues are negative and a repellor when both are positive. Trajectories spiral toward the attractor or away from the repellor when the eigenvalues are complex, but move straight in or out when they are real. A *saddle* occurs when one eigenvalue is positive and the other is negative. Two trajectories, called the *stable manifold*, approach the saddle asymptotically, two others, called the *unstable manifold*, approach the saddle asymptotically in reverse time, and all other nearby trajectories approach then move away. Figure 1 illustrates these behaviors. The Jacobian provides no information when one or both eigenvalues have real part zero. I discuss this case in the final section.

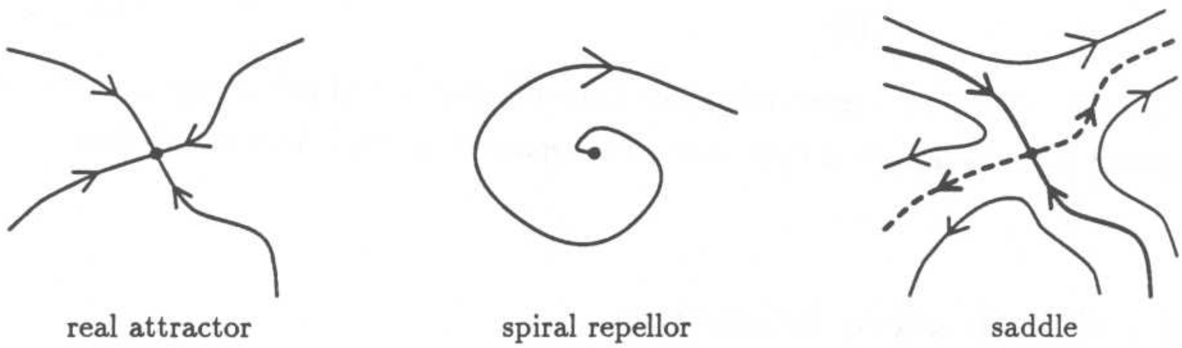
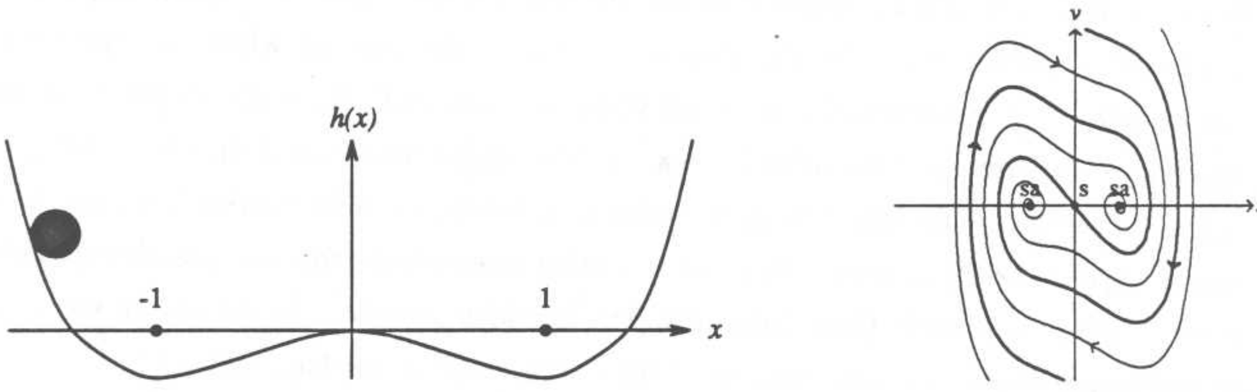


Figure 1: Sample fixed points. The stable and unstable manifolds of the saddle appear as solid and broken thick lines.

A *phase diagram* for a system depicts its phase space and prototypical trajectories. For example, the system

$$\dot{x} = v$$
$$\dot{v} = -\mu v - h'(x)$$

models the motion of a ball on a track. The variables x and v represent location and velocity, $h(x)$ is the height of the track, and μ is the damping coefficient. Figure 2 shows the ball on the track and the phase diagram of its model for $h(x) = x^4/4 - x^2/2$ and $\mu = 1$. PHASER generated this and all subsequent phase diagrams. The local maximum of h at 0 produces the saddle at $(0,0)$; the local minima at ± 1 produce the spiral attractors. The two trajectories in the stable manifold of $(0,0)$ represent cases where the ball comes to rest at $x = 0$. They divide phase space into two regions, corresponding to cases where the ball comes to rest at $x = -1$ and $x = 1$.



(a) (b)

Figure 2: (a) Ball on a track and (b) its phase diagram. Fixed points appear as solid circles labeled with the initials of their types. The separatrix is marked with thick lines.

3 Qualitative behavior

A qualitative description of a system consists of a partition of its phase space into connected sets of qualitatively equivalent states. The equivalence criterion depends on the problem task. In this paper, I take the standard approach of treating two states as equivalent if their corresponding trajectories have the same asymptotic behavior. The Poincaré-Bendixson theorem [3, Ch. 11] provides a complete characterization of asymptotic behavior in the plane. Every bounded trajectory asymptotically approaches either a fixed point, a limit cycle, or a closed path consisting of saddles and their stable manifolds, called a *separatrix* (Fig. 3).

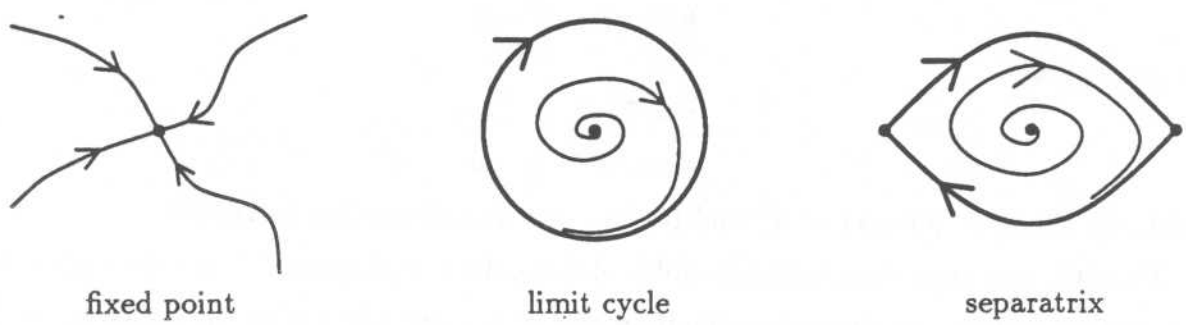


Figure 3: Bounded asymptotic behaviors in the plane.

Limit cycles, separatrices, and the unstable manifolds of saddles behave differently from all their neighboring trajectories. They are singleton equivalence classes and closed sets. The equivalence classes of the other trajectories are open sets. Any such trajectory, $\phi(t)$, leaves the bounding box or approaches an attractor as t approaches $\pm\infty$ by the Poincaré-Bendixson theorem. Each admissible point on ϕ has a neighborhood with the same asymptotic behaviors at $\pm\infty$ because of the continuous dependence of a trajectory on its initial state [3, Ch. 8]. (I ignore the rare case of a tangential intersection between the trajectory and the bounding box.) The closed equivalence classes form the boundaries between the open ones. In the track example, the separatrix delimits the open equivalence classes of the two attractors.

4 The partition algorithm

PHASER partitions the phase space of a system into open equivalence classes and boundary trajectories. It forms an initial partition whose sole region consists of the bounding box, then iteratively adds boundary trajectories until each region consists entirely of equivalent states.

It organizes the search for boundary trajectories around the fixed points of the system, which it calculates by setting all derivatives to zero and solving the resulting equations. The current implementation uses a simple symbolic equation solver, which suffices for all the examples in this paper. In other work [4], I describe a more powerful equation solver that combines numeric and symbolic techniques. For example, given the Lienard equation

$$\dot{x} = y$$
$$\dot{y} = -x - x^2 - y$$

PHASER solves

$$y = 0$$
$$-x - x^2 - y = 0$$

to obtain the fixed points $(-1, 0)$ and $(0, 0)$, a saddle and a spiral attractor.

PHASER generates the stable manifolds of the saddles and groups them into separatrices. The two trajectories in the stable manifold of a saddle are tangent to the eigenvector of its negative eigenvalue [2, Ch. 1]. PHASER approximates them by simulating backward from two nearby points on the eigenvector, one on each side of the saddle. In our example, it forms a single separatrix from $(-1, 0)$ and its stable manifold (Fig. 4a). Next, it generates the unstable manifolds of the saddles by simulating forward from points on the eigenvectors of their positive eigenvalues. In our example, it finds connections from $(-1, 0)$ to $(0, 0)$ and to the bounding box (Fig. 4b). It has now partitioned the bounding box into three regions (Fig. 5). Trajectories in the middle region approach the spiral attractor $(0, 0)$; the rest leave the bounding box.

The remaining boundary trajectories are limit cycles. Limit cycles can only exist in regions that contain fixed points because they each must enclose at least one fixed point other than a saddle [2]. All trajectories in other regions must leave the bounding box by the Poincaré-Bendixson theorem. PHASER can sometimes rule out limit cycles by theoretical means. Examples are Hamiltonian systems (discussed in the final section) and systems whose divergence, $\partial f/\partial x + \partial g/\partial y$, is strictly positive or strictly negative. In the remaining cases, PHASER scans straight upward from each non-saddle fixed point until it finds all enclosing limit cycles within the bounding box.

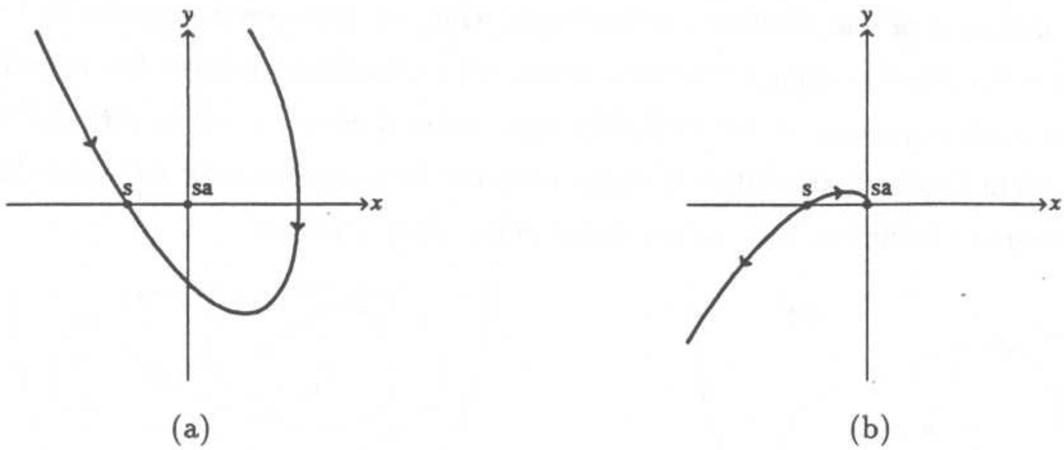


Figure 4: (a) Stable manifold and (b) unstable manifold of the Lienard equation.

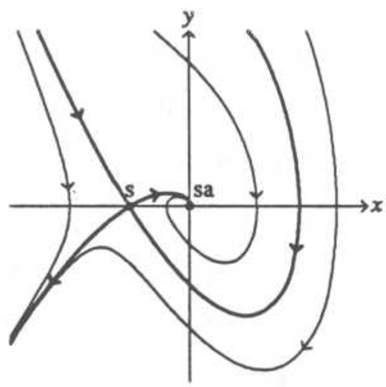
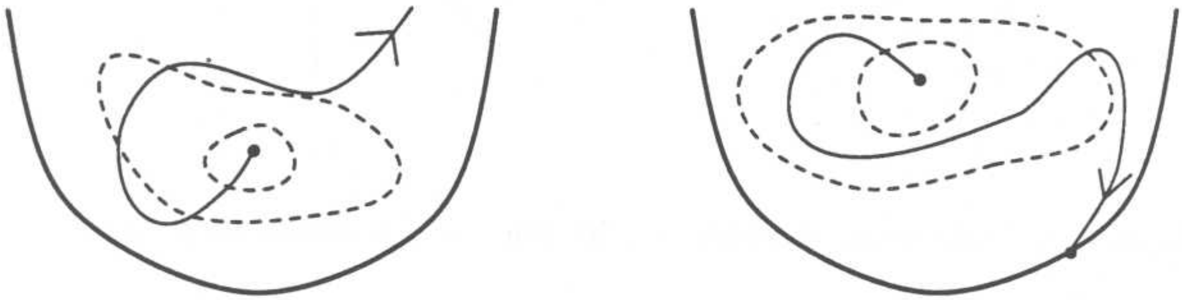


Figure 5: PHASER's partition of the region $|x|, |y| < 3$ in the phase space of the Lienard equation.

First consider a repellor. PHASER generates the trajectory through a point slightly above it. If the trajectory leaves the bounding box or approaches a saddle on the region's boundary, it intersects every curve that encloses the fixed point (Fig. 6). The region contains no limit cycles because trajectories cannot intersect, hence all its trajectories leave the bounding box. By the same argument, if the trajectory approaches a closed curve (a separatrix or limit cycle), as in Figure 3, all contained trajectories do the same. PHASER only adds limit cycles to its current partition, since all separatrices are already known.



(a) (b)

Figure 6: Trajectories that (a) leave the bounding box or (b) approach a saddle intersect all curves in the region that enclose the fixed point.

PHASER handles attractors analogously. It generates and examines a time-reversed trajectory instead of a forward trajectory. The conclusions about limit cycles are the same, but the asymptotic behavior is the reverse of before: trajectories approach the attractor rather than leaving it. In the Lienard example, the time-reversed trajectory for $(0,0)$ leaves the bounding box (Fig. 5), implying that all trajectories in the region approach this attractor.

After hitting a closed curve, the remainder of the search for limit cycles is the same for all fixed points. PHASER generates the trajectory through a point directly above the fixed point and just outside the closed curve. It halts if the trajectory leaves the bounding box or approaches a saddle. There are no more limit cycles and all other trajectories outside the closed curve leave the bounding box. If the trajectory approaches another closed curve, PHASER records the new curve and repeats this process. All trajectories in the band between the two curves approach the new curve. If the trajectory approaches the previous closed curve, PHASER tests the time-reversed trajectory. The analysis is identical with the conclusions about asymptotic behavior reversed. Figure 7 illustrates these possibilities.

(a) (b)

Figure 7: Testing for additional limit cycles: (a) found outer limit cycle (b) done.

PHASER need not search for limit cycles in the Lienard and track examples because both systems have negative divergence. The qualitative descriptions shown in Figures 2 and 5 are complete. The Van der Pol equation

$$\dot{x} = y$$
$$\dot{y} = -x - 0.5y(x^2 - 1)$$

provides a nontrivial example. The system has a single, repelling fixed point at $(0, 0)$, hence no separatrices. PHASER finds an attracting limit cycle that all trajectories in the bounding box $|x|, |y| < 4$ approach (Fig. 8).

The model of aeroelastic galloping of a square prism in a steady wind

$$\dot{x} = y$$
$$\dot{y} = -kx - ry + \frac{\rho}{2}V^2 a \left(A_1 \frac{y}{V} - A_3 \frac{y^3}{V^3} + A_5 \frac{y^5}{V^5} - A_7 \frac{y^7}{V^7} \right)$$

provides an example of multiple limit cycles. PHASER's phase diagrams agree with those of Thompson and Stewart [6] for all their choices of parameter values. One choice yields three concentric limit cycles enclosing a spiral repellor at $(0, 0)$, as shown in Figure 9. The inner and outer limit cycles are attractors, whereas the middle one is a repellor.

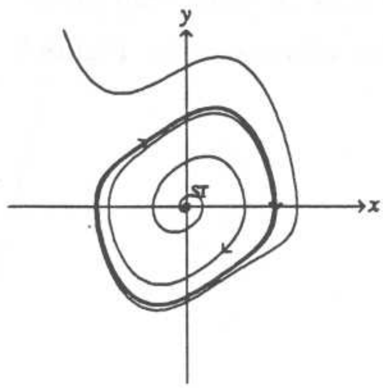


Figure 8: Phase diagram of the Van der Pol equation.

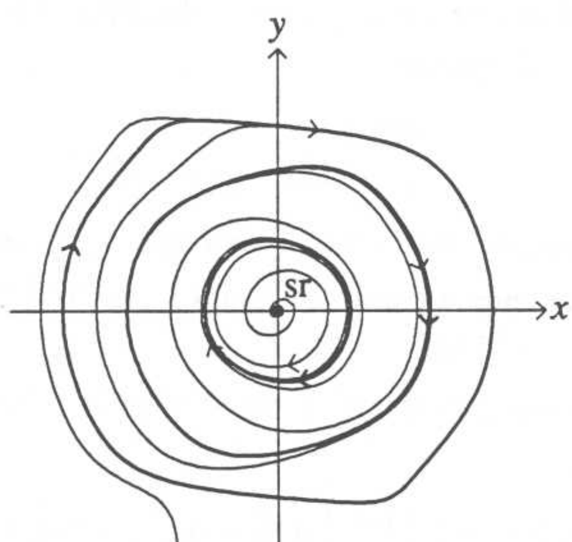


Figure 9: Phase diagram of the aeroelastic galloping model.

5 Robust numeric simulation

PHASER calculates trajectories numerically by the fourth-order Runge-Kutta algorithm with variable step size; any other integration algorithm can be substituted transparently. It links the simulation data into trajectories with smooth curves. It modifies these trajectories in light of the uniqueness of solutions, which implies that trajectories never intersect. If a newly constructed trajectory intersects an existing one, it merges the segments of the trajectories closer to the initial points and discards the further segments (Fig. 10). It converts self-intersecting trajectories into cycles analogously (Fig. 11). It measures all distances to within a user specified tolerance (10^{-3} in this paper), permitting it to recognize intersections despite small numeric errors. It uses the same tolerance in identifying asymptotic approach toward fixed points, limit cycles, and separatrices.

Figure 10: Intersecting trajectories and their interpretation.

Figure 11: A self-intersecting trajectory and its interpretation.

6 Conclusions

This paper describes PHASER, a program that partitions the phase space of a system of two ordinary differential equations into regions of asymptotically equivalent trajectories.

The boundaries between regions are limit cycles, separatrices, and the unstable manifolds of saddles. PHASER detects these boundaries by numeric simulation, guided by knowledge about dynamics. It determines the behavior near fixed points from the Jacobian of the system, scans outward to find separatrices and limit cycles, and stops upon reaching the boundary of a prespecified region of interest. It describes the asymptotic behavior of the system in terms of the qualitative properties of the partition, abstracting away the numeric details of the boundary trajectories.

The theory of local behavior breaks down at *nonhyperbolic* fixed points where the Jacobian has one or more eigenvalues with zero real parts. In the general case, PHASER must determine the local behavior by enclosing the nonhyperbolic point in a small circle and generating trajectories through many points on its perimeter. (It could also direct the search according to the center manifold theorem [2].) It can determine the local behavior of some honhyperbolic points by theoretical means, rendering search unnecessary. For example, those of *Hamiltonian* systems

$$\dot{x} = \partial H(x, y)/\partial y$$
$$\dot{y} = -\partial H(x, y)/\partial x$$

are typically surrounded by periodic trajectories. PHASER currently recognizes Hamiltonian systems of the form $H(x, y) = f(x) + g(y)$, which include the un[illegible]ndulum, $\ddot{x} = \sin x$, and the frictionless version ($\mu = 0$) of the track example. Straightforward algorithms exist for recognizing additional Hamiltonian systems.

Automating the choice of a bounding box and of a simulation tolerance is a topic for future research. One possibility, suggested to me by Eric Horvitz, is to weigh simulation cost against the cost of erroneous or incomplete results, using decision theoretic techniques. Another project is to strengthen PHASER's ability to establish or rule out specific behaviors by theoretical means, thus reducing simulation effort and increasing the reliability of its conclusions. One could also extend PHASER to nonplanar phase spaces, such as cylinders and tori, which arise naturally in physical applications. I describe steps in this direction elsewhere [5].

The biggest challenges are bifurcation analysis and analysis of larger systems. Most models contain parameters. One must partition the parameter space into regions with similar phase diagrams separated by *bifurcation curves* on which the phase diagram changes. PHASER can handle the simplest case where a fixed point changes type at a certain parameter value. The theory of fixed point bifurcation extends to limit cycles, but does not provide bifurcation values explicitly. The work of Abelson [1] should be relevant here. Other, *global* bifurcations remain to be studied.

Larger systems are vastly more complicated than second-order ones. The Poincaré-Bendixson theorem breaks down, leading to a rich collection of additional asymptotic behaviors, including chaotic ones. Few tools exist for analyzing these behaviors and the challenges are great. I plan to start with the simplest case: systems of two forced ordinary differential equations. Yip's work [7] in conservative discrete dynamic systems should be relevant here.

Acknowledgements

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